

New area-minimizing Lawson-Osserman cones

Xiaowei Xu Ling Yang Yongsheng Zhang

October 27, 2016

Abstract: In this paper we shall study three types of Lawson-Osserman cones which were introduced in [XYZ]. They are compositions of a Hopf fibration and standard isometric minimal immersions of degree 2 (see §3). By virtue of Lawlor's criterion in [Law91], we are able to show that they are area-minimizing. In particular, two undetermined minimal cones given in [LO77] will be shown area-minimizing.

Keywords: area-minimizing, Hopf fibration, Lawson-Osserman cone.

Mathematics Subject Classification(2010): 53A10, 58E20.

Contents

1	Introduction	1
2	Quaternions, octonions and odd-dimensional spheres	3
3	On minimal spheres of Lawson-Osserman type	4
3.1	Type-I minimal spheres	5
3.2	Type-II minimal spheres	8
3.3	Type-III minimal spheres	12
4	On the area-minimizing property	14
4.1	Lawlor's criterion	14
4.2	Proof for the area-minimizing property	16

1 Introduction

A well-known result of Federer (Theorem 5.4.3 in [Fed69], also see Theorem 35.1 and Remark 34.6 (2) in Simon [Sim83]) states that a tangent cone at a point of an area-minimizing rectifiable current is itself area-minimizing. Therefore, area-minimizing cones reflect local behaviors of area-minimizing rectifiable currents. Accordingly, the study about them remains as a central topic in the geometric measure theory.

A lot of work has been done for area-minimizing hypercones, e.g. [Fle62, DG65, Alm66, Sim68, BDGG69, Law72, Sim74, Sim73, HS85, FK85, Sim83, Law91]. Following [Law72], [Che88]

found homogeneous area-minimizing cones of codimension two. Around the same time, Lawlor introduced a systematic method, called the curvature criterion, in [Law91] which can determine many cones' being area-minimizing, for instance, the classification of area-minimizing cones over products of spheres and the first minimizing cones over unorientable surfaces in the sense of mod 2. Thereafter, he gained more with Kerckhove in [KL99].

Among others, an important minimal cone given by [LO77] and shown area-minimizing in [HL82] is the following. Let η , η' and η'' denote the Hopf maps $S^{2^i-1}(1) \rightarrow S^{2^{i-1}-1}(1)$ for $i = 2, 3$ and 4 respectively. Then Lawson and Osserman considered cones over images of F , F' and F'' given by

$$S^{2^i-1}(1) \rightarrow S^{2^i+2^{i-1}-1}(1), \quad x \mapsto (\alpha_2 x, \beta_2 \eta(x)), (\alpha_3 x, \beta_3 \eta'(x)), (\alpha_4 x, \beta_4 \eta''(x))$$

respectively, with suitable constants α_i and β_i . The resulting cones become minimal and therefore represent Lipschitz (not C^1) solutions for Dirichlet problem of minimal surfaces for boundary data

$$\phi = \frac{\beta_2}{\alpha_2} \eta, \quad \phi' = \frac{\beta_3}{\alpha_3} \eta', \quad \phi'' = \frac{\beta_4}{\alpha_4} \eta''.$$

It was shown later in [HL82] that the cone generated by η is calibrated by a canonical coassociative 4-form in \mathbf{R}^7 and, hence, area-minimizing by the fundamental theorem of calibrated geometry. The calibration form in some way exhibits a special interaction between algebraic and geometric structures of \mathbf{R}^7 . Due to the lack of similar understandings, it remained open whether the other two cones are area-minimizing.

In this paper, we can answer both affirmatively. In fact we establish more. Inspired by [LO77], we constructed in [XYZ] minimal cones of Lawson-Osserman type which are homeomorphic to Euclidean spaces. They are compositions of a Hopf fibration and an isometric minimal immersion into spheres. Here, by regarding identity maps (up to $\sqrt{2}$ -scaling and a rigidity) between spheres as isometric minimal immersions, we include cones corresponding to η , η' and η'' . Because of huge moduli spaces for the second component (see, for example, [dCW71, Tot97, Ura85]), our construction produces quite a lot of series of uncountably many examples. We have observed that some of them are non-minimizing due to the amusing spiral behaviors under a peculiar dynamic system in [XYZ]. In this paper we shall show, in the opposite direction, that all the three types with the second slot coming from immersions of degree 2, including the other undetermined cones induced by η' and η'' , are area-minimizing. In particular, our results indicate that every topological space \mathbf{R}^{2k} for $k > 1$ can bend to be a nontrivial area-minimizing cone in certain \mathbf{R}^N .

The paper is organized as follows. We introduce notations in §2 for calculations. In §3, three types of Lawson-Osserman cones are mentioned and standard isometric minimal immersions of degree 2 are explained. Since our examples are equivariant, we analyze second fundamental forms only at a base point in §3.1-3.3 for each type respectively. By explanations on Lawlor's curvature criterion in §4.1 and the computation results in §3, we show in §4.2 that vanishing angles exist for all of them except the one given by η and that the corresponding normal wedges do not intersect. Hence, by [HL82] and Lawlor's curvature criterion, they are all area-minimizing.

2 Quaternions, octonions and odd-dimensional spheres

We recall some basic facts about quaternions and octonions and make a choice of preferred local parameterization of odd-dimensional spheres at a fixed point by exponential map.

Let \mathbf{R} , \mathbf{C} be the real and complex number fields, and \mathbf{R}^n , \mathbf{C}^n the real and complex n -tuple spaces respectively. Naturally, we identify \mathbf{C}^n with \mathbf{R}^{2n} by $(z_1, \dots, z_n) \mapsto (x_1, \dots, x_n, x_{n+1}, \dots, x_{2n})$, where $z_k = x_k + i x_{n+k} \in \mathbf{C}$ and $i^2 = -1$. Let \mathbf{H} be the real division algebra of quaternions. An element of \mathbf{H} can be written uniquely as $a = z_1 + z_2 j$, where $z_1, z_2 \in \mathbf{C}$ and $j \in \mathbf{H}$ satisfies $j^2 = -1$, $zj = j\bar{z}$, for all $z \in \mathbf{C}$. In this way, \mathbf{C} sits in \mathbf{H} as a subalgebra and \mathbf{H} becomes a complex vector space under \mathbf{C} -action on the left. Thus, the n -tuples \mathbf{H}^n can be identified with \mathbf{C}^{2n} by $(a_1, \dots, a_n) \mapsto (z_1, z_2, \dots, z_{2n-1}, z_{2n})$, where $a_k = z_{2k-1} + z_{2k} j \in \mathbf{H}$. Let \mathbf{O} be the algebra of octonions. The elements in \mathbf{O} can be written as $p = a_1 + a_2 e$, where $a_1, a_2 \in \mathbf{H}$, $e \in \mathbf{O}$ and $e^2 = -1$, $pe = -a_2 + a_1 e$. For another $q = a_3 + a_4 e \in \mathbf{O}$, we have $pq = (a_1 a_3 - \bar{a}_4 a_2) + (a_4 a_1 + a_2 \bar{a}_3) e$. Similarly, the n -tuples \mathbf{O}^n can be identified with \mathbf{H}^{2n} by $(p_1, \dots, p_n) \mapsto (a_1, a_2, \dots, a_{2n-1}, a_{2n})$, where $p_k = a_{2k-1} + a_{2k} e \in \mathbf{O}$. The conjugate of a quaternion (resp. octonion) is defined by $\bar{a} = \bar{z}_1 - z_2 j$ (resp. $\bar{p} = \bar{a}_1 - a_2 e$). Hence, the norm of a quaternion (resp. octonion) is given by $|a|^2 = a\bar{a}$ (resp. $|p|^2 = p\bar{p}$).

An odd-dimensional unit sphere can be described by

$$S^{2m+1}(1) = \left\{ (z_1, \dots, z_{m+1}) \in \mathbf{C}^{m+1} \mid \sum_{k=1}^{m+1} |z_k|^2 = 1 \right\}.$$

For a local parameterization at point $p = (1, 0, \dots, 0) \in S^{2m+1}$, we set

$$X_1 = i(E_{11} - E_{22}), \quad X_k = E_{1k} - E_{k1}, \quad X_{m+k} = i(E_{1k} + E_{k1}),$$

where $2 \leq k \leq m+1$, E_{kl} is the $(m+1) \times (m+1)$ -matrix with value 1 in the (k, l) -slot and zero for others. Then, it is easy to check that

$$p(t) = p e^{t_1 X_1} e^{t_2 X_2} \dots e^{t_{2m+1} X_{2m+1}} \quad \text{for small } t = (t_1, \dots, t_{2m+1}) \in \mathbf{R}^{2m+1}$$

parameterizes some neighborhood of p . Writing $p(t) = (z_1, \dots, z_{m+1})$, through direct calculations, we have

$$z_1 = \left(1 - \frac{1}{2} \sum_{A=1}^{2m+1} t_A^2\right) + i \left(t_1 + \sum_{k=2}^{m+1} t_k t_{m+k}\right) + o(t^2), \quad (2.1)$$

$$z_k = (t_k - t_1 t_{m+k}) + i(t_{m+k} + t_1 t_k) + o(t^2), \quad (2.2)$$

where $2 \leq k \leq m+1$ and $t^2 = \sum_{A=1}^{2m+1} t_A^2$. In our parameterization, the tangent space $T_p S^{2m+1}$ is

spanned by $\{\epsilon_A = \frac{\partial p(t)}{\partial t_A} \Big|_{t=0}\}$, more explicitly, by

$$\begin{aligned} \epsilon_1 &= iE_1 \in \mathbf{C}^{m+1}, & \text{or} & & (0, E_1) &\in \mathbf{R}^{2m+2}, \\ \epsilon_k &= E_k \in \mathbf{C}^{m+1}, & \text{or} & & (E_k, 0) &\in \mathbf{R}^{2m+2}, \\ \epsilon_{m+k} &= iE_k \in \mathbf{C}^{m+1}, & \text{or} & & (0, E_k) &\in \mathbf{R}^{2m+2}, \end{aligned}$$

where $2 \leq k \leq m+1$ and E_k is the vector in \mathbf{R}^{m+1} with value 1 in the k -th position and zero for others.

3 On minimal spheres of Lawson-Osserman type

In this section we introduce what we are concerned with in this paper. They are embedded minimal spheres given by composition of a Hopf fibration and minimal immersions from complex projective spaces, quaternion projective spaces or the Cayley projective line into unit spheres. The idea is inspired by Lawson and Osserman's original construction in [LO77]. So, we call such minimal spheres of **Lawson-Osserman type** and the associated cones **Lawson-Osserman cones**. Since there are only three Hopf fibrations $\pi^{2n+1,2n} : S^{2n+1} \longrightarrow \mathbf{CP}^n$, $\pi^{4n+3,4n} : S^{4n+3} \longrightarrow \mathbf{HP}^n$ and $\pi^{15,8} : S^{15} \longrightarrow \mathbf{OP}^1$, we divide minimal spheres of Lawson-Osserman type into three types. It is worth mentioning Z.Z.Tang [Tan01] proved there is no submersion from S^{23} to the Cayley projective plane.

Let us first recall the classical Lawson-Osserman construction. By the Hopf map $\eta : S^3 \longrightarrow S^2$, $(z_1, z_2) \mapsto (|z_1|^2 - |z_2|^2, 2\bar{z}_1 z_2)$, Lawson and Osserman [LO77] gave a minimal immersion

$$F : S^3(1) \longrightarrow S^6(1), \quad z \mapsto \left(\frac{2}{3}z, \frac{\sqrt{5}}{3}\eta(z) \right).$$

Later, in [HL82], Harvey and Lawson proved that the cone of F is area-minimizing. Therefore, the topological space \mathbf{R}^4 can emerge in \mathbf{R}^7 as a nontrivial area-minimizing cone. To find more such minimal immersions, we observe η from another perspective. In fact, it is a composition of the Hopf fibration $\pi^{3,2}$ and a degree 2 map Φ from \mathbf{CP}^1 into S^2 . Explicitly,

$$\pi^{3,2} : S^3 \longrightarrow \mathbf{CP}^1, (z_1, z_2) \mapsto [z_1, z_2], \text{ and } \Phi : \mathbf{CP}^1 \longrightarrow S^2, [z_1, z_2] \mapsto (|z_1|^2 - |z_2|^2, 2\bar{z}_1 z_2).$$

Analogous constructions in [LO77] were also given for Hopf maps $\eta' : S^7 \longrightarrow S^4$, $(a_1, a_2) \mapsto (|a_1|^2 - |a_2|^2, 2\bar{a}_1 a_2)$ and $\eta'' : S^{15} \longrightarrow S^8$, $(p_1, p_2) \mapsto (|p_1|^2 - |p_2|^2, 2\bar{p}_1 p_2)$. However, to our knowledge, it was unknown whether the cones associated to η' and η'' are area-minimizing. By results to be established in this section, we can have positive conclusions for both in §4.

By composing Hopf fibration and isometric minimal immersions of degree 2 (explained below) from \mathbf{CP}^n , \mathbf{HP}^n into unit spheres, we gain lots of minimal spheres of Lawson-Osserman type. The minimality follows from a general theorem of authors [XYZ]. Notice that \mathbf{CP}^n , \mathbf{HP}^n and unit spheres are compact symmetric space. Hence, such immersions can be realized by equivariant ones compatible with their Lie group structure.

Now, we review standard minimal immersions of a compact irreducible Riemannian symmetric space (M, g) into unit spheres (see [dCW71, Ura85]). Let Δ be the Laplace-Beltrami operator of (M, g) acting on C^∞ -functions, λ_k the k -th eigenvalue of Δ with $0 = \lambda_0 < \lambda_1 < \dots$, and V^k the corresponding eigenspace. Set $\dim V^k = m(k) + 1$. Then one can define an L^2 -inner product on V^k by

$$(f, h) := \int_M f h d\mu$$

where $d\mu$ of (M, g) is the normalized canonical measure with $\int_M d\mu = m(k) + 1$. Suppose $\{f_0, \dots, f_{m(k)}\}$ form an orthonormal basis of V^k . By Takahashi's Theorem [Tak66], the standard map x_k from M into $\mathbf{R}^{m(k)+1}$ by sending $p \in M$ to $(f_0(p), \dots, f_{m(k)}(p))$ gives an isometric minimal immersion of $(M, \frac{\lambda_k}{\dim M} g)$ into $S^{m(k)}(1)$. This standard minimal immersion x_k can also understood as follows. Let (G, K) be a symmetric pair with $M = G/K$. Then, a point of M can

be viewed as σK for some $\sigma \in G$, and G acts on V^k by $(\sigma \cdot f)(p) = f(\sigma^{-1}p)$, $\sigma \in G$, $p \in M$. In this way, an orthogonal representation of G is given in terms of $\sigma \cdot f_\alpha = \sum_{\alpha=0}^{m(k)} a_{\alpha\beta} f_\beta$ by

$$\rho_k : G \longrightarrow O(m(k) + 1), \quad \sigma \mapsto (a_{\alpha\beta}(\sigma)). \quad (3.1)$$

Up to a rigidity of V^k , one can assume that $(f_1(eK), \dots, f_{m(k)}(eK)) = E_1 = (1, 0, \dots, 0)$. Then we get $x_k(\sigma K) = E_1 \rho_k(\sigma)$, $\sigma \in G$.

We give an alternative description on V^k for $M = \mathbf{CP}^n$ equipped with the Fubini-Study metric. Let ϕ be a complex valued homogeneous polynomial in $2n+2$ variables $z_1, \dots, z_{n+1}, \bar{z}_1, \dots, \bar{z}_{n+1}$ on \mathbf{C}^{n+1} . It is said to be of (p, q) -type if

$$\phi(cz_1, \dots, cz_{n+1}, \bar{c}\bar{z}_1, \dots, \bar{c}\bar{z}_{n+1}) = c^p \bar{c}^q \phi(z_1, \dots, z_{n+1}, \bar{z}_1, \dots, \bar{z}_{n+1}), \text{ for } \forall c \in \mathbf{C}.$$

Denote by $P_{p,q}^{n+1}$ the complex vector space of all homogeneous polynomials of (p, q) -type on \mathbf{C}^{n+1} . Note that functions in $P_{q,q}^{n+1}$ are S^1 -invariant. So, they descend to functions on \mathbf{CP}^n . Set

$$H_{p,q}^{n+1} = \{ \phi \in P_{p,q}^{n+1} \mid D\phi = 0 \}, \text{ where } D = -4 \sum_{k=1}^{n+1} \frac{\partial^2}{\partial z_k \partial \bar{z}_k}.$$

It is a well-known fact (see [Mas80, Ura85]) that the k -th eigenspace V^k is $SU(n+1)$ -isomorphism to $H_{k,k}^{n+1} \cap C^\infty(\mathbf{C}^{n+1}, \mathbf{R})$ for \mathbf{CP}^n via the Hopf fibration $\pi^{2n+1, 2n}$, where $C^\infty(\mathbf{C}^{n+1}, \mathbf{R})$ consists of all real valued C^∞ -functions on \mathbf{C}^{n+1} . In present paper, we shall focus on $H_{1,1}^{n+1} \cap C^\infty(\mathbf{C}^{n+1}, \mathbf{R})$, the space of isometric minimal immersions of degree 2. There is also a similar description in terms of quaternions valid for \mathbf{HP}^n equipped with the canonical metric.

Throughout this paper, we will, if not otherwise specified, use the following convention for indices:

$$1 \leq A, B, \dots \leq 2m+1; \quad 2 \leq k, l, \dots \leq n+1; \quad 2 \leq \alpha, \beta, \dots \leq n,$$

where m will take n or $2n+1$ in the sequel.

3.1 Type-I minimal spheres

To obtain a standard minimal immersion from \mathbf{CP}^n into unit sphere, we need to find an orthonormal basis of $H_{1,1}^{n+1} \cap C^\infty(\mathbf{C}^{n+1}, \mathbf{R})$. For an element $\phi \in H_{1,1}^{n+1} \cap C^\infty(\mathbf{C}^{n+1}, \mathbf{R})$, it follows

that $\phi = \sum_{k,l=1}^{n+1} \lambda_{k\bar{l}} z_k \bar{z}_l$ subject to $D\phi = 0$ and $\bar{\phi} = \phi$. Set $\lambda_{l\bar{k}} = \bar{\lambda}_{k\bar{l}}$. The requirements become

$$\sum_{k=1}^{n+1} \lambda_{k\bar{k}} = 0, \quad \lambda_{k\bar{l}} = \lambda_{l\bar{k}}. \quad (3.2)$$

With this understanding, we have

Lemma 3.1. *There is an orthonormal basis of $H_{1,1}^{n+1} \cap C^\infty(\mathbf{C}^{n+1}, \mathbf{R})$ w.r.t. the L^2 -inner product given by*

$$\phi_\alpha = c_{n,\alpha} \left(|z_\alpha|^2 - \frac{1}{n+1-\alpha} \sum_{k=\alpha+1}^{n+1} |z_k|^2 \right), \quad 1 \leq \alpha \leq n,$$

$$\phi_{kl} = d_n \operatorname{Re}(z_k \bar{z}_l), \quad \phi_{\bar{k}\bar{l}} = d_n \operatorname{Im}(z_k \bar{z}_l), \quad 1 \leq k < l \leq n+1,$$

where $c_{n,\alpha} = \sqrt{\frac{(n+1)(n+1-\alpha)}{n(n+2-\alpha)}}$ and $d_n = \sqrt{\frac{2(n+1)}{n}}$.

Proof. From (3.2), we know that $u_\alpha = |z_\alpha|^2 - |z_{\alpha+1}|^2$, $v_{kl} = \operatorname{Re}(z_k \bar{z}_l)$ and $v_{\bar{k}\bar{l}} = \operatorname{Im}(z_k \bar{z}_l)$ form a basis of $H_{1,1}^{n+1} \cap C^\infty(\mathbf{C}^{n+1}, \mathbf{R})$. Then, following the Schmidt orthonormalization process w.r.t. the L^2 -inner product, one can get $\{\phi_\alpha, \phi_{kl}, \phi_{\bar{k}\bar{l}}\}$ as an orthonormal basis of $H_{1,1}^{n+1} \cap C^\infty(\mathbf{C}^{n+1}, \mathbf{R})$. We leave details to readers. \square

Hence, we gain an isometric minimal immersion $\Phi : \mathbf{CP}^n \longrightarrow S^{n(n+2)-1}(1)$ expressed by $[z] \mapsto (\phi_\alpha(z), \phi_{kl}(z), \phi_{\bar{k}\bar{l}}(z))$ and a Lawson-Osserman sphere given by $f = \Phi \circ \pi^{2n+1, 2n} : S^{2n+1}(1) \rightarrow S^{n(n+2)-1}(1)$. In particular, f is just the Hopf map η when $n = 1$. The type-I Lawson-Osserman minimal sphere is represented by

$$F : S^{2n+1}(1) \longrightarrow S^{(n+1)^2+2n}(1), \quad z \mapsto (a_n(\operatorname{Re}(z), \operatorname{Im}(z)), b_n(\phi_\alpha(z), \phi_{kl}(z), \phi_{\bar{k}\bar{l}}(z))),$$

where $a_n = \sqrt{\frac{2(n+1)}{(2n+1)(n+2)}}$ and $b_n = \sqrt{\frac{n(2n+3)}{(2n+1)(n+2)}}$.

Remarks. (a) For coefficients a_n and b_n which uniquely determine the minimality we refer to our recent paper [XYZ]. We will just check the minimality property in Proposition 3.2;

(b) By the construction of Φ , F is in fact homogeneous and its image is an $\operatorname{Id} \oplus \rho_1(SU(n+1))$ -orbit through the base point

$$P = (a_n E_1, 0, b_n E_1, 0, 0),$$

where ρ_1 is defined in (3.1). Therefore, we will study its geometry only around one point for our purpose.

More precisely, we shall compute the second fundamental form of F at P in the remaining part of this subsection. Substituting (2.1) and (2.2) for $m = n$ into expressions of $\phi_\alpha, \phi_{kl}, \phi_{\bar{k}\bar{l}}$, we obtain

$$\begin{aligned} \phi_1 &= 1 - \frac{n+1}{n} \sum_{A=2}^{2n+1} t_A^2 + o(t^2), \\ \phi_\alpha &= c_{n,\alpha} \left[t_\alpha^2 + t_{n+\alpha}^2 - \frac{1}{n+1-\alpha} \sum_{k=\alpha+1}^{n+1} (t_k^2 + t_{n+k}^2) \right] + o(t^2), \\ \phi_{1k} &= d_n t_k + o(t^2), \quad \phi_{1\bar{k}} = -d_n t_{n+k} + o(t^2), \\ \phi_{kl} &= d_n (t_k t_l + t_{n+k} t_{n+l}) + o(t^2), \\ \phi_{\bar{k}\bar{l}} &= d_n (-t_k t_{n+l} + t_l t_{n+k}) + o(t^2), \end{aligned} \tag{3.3}$$

where $2 \leq \alpha \leq n$, $2 \leq k \leq n+1$ and $2 \leq k < l \leq n+1$. Noticing that $F_*(\varepsilon_A) = \frac{\partial F}{\partial t_A} \Big|_{t=0}$, by (3.3),

(2.1) and (2.2), we have

$$\begin{aligned} F_*(\varepsilon_1) &= (0, a_n E_1, 0, 0, 0), \\ F_*(\varepsilon_k) &= (a_n E_k, 0, 0, b_n d_n E_{1k}, 0), \\ F_*(\varepsilon_{n+k}) &= (0, a_n E_k, 0, 0, -b_n d_n E_{1k}), \end{aligned}$$

where $2 \leq k \leq n+1$. By normalization, we get an orthonormal basis of $F_*(T_p S^{2n+1})$:

$$e_1 = \frac{1}{a_n} F_*(\varepsilon_1), \quad e_k = \frac{1}{\sqrt{a_n^2 + b_n^2 d_n^2}} F_*(\varepsilon_k), \quad e_{n+k} = \frac{1}{\sqrt{a_n^2 + b_n^2 d_n^2}} F_*(\varepsilon_{n+k});$$

and an orthonormal basis for the normal space of $F_*(T_p S^{2n+1})$ in $T_p S^{(n+1)^2+2n}$:

$$\begin{aligned} e_{2n+2} &= (-b_n E_1, 0, a_n E_1, 0, 0), \\ e_{2n+1+k} &= \frac{1}{\sqrt{a_n^2 + b_n^2 d_n^2}} (-b_n d_n E_k, 0, 0, a_n E_{1k}, 0), \\ e_{3n+1+k} &= \frac{1}{\sqrt{a_n^2 + b_n^2 d_n^2}} (0, b_n d_n E_k, 0, 0, a_n E_{1k}), \\ e_{4n+1+\alpha} &= (0, 0, E_\alpha, 0, 0), \\ e_{kl} &= (0, 0, 0, E_{kl}, 0), \\ e_{\bar{k}\bar{l}} &= (0, 0, 0, 0, E_{kl}), \end{aligned} \tag{3.4}$$

where $2 \leq k \leq n+1$, $2 \leq \alpha \leq n$ and $2 \leq k < l \leq n+1$.

Set $F_{AB} = \frac{\partial^2 F}{\partial t_A \partial t_B} \Big|_{t=0}$. Through direct computations, we obtain

$$\begin{aligned} F_{11} &= (-a_n E_1, 0, 0, 0, 0), \\ F_{\alpha\alpha} = F_{n+\alpha n+\alpha} &= \left(-a_n E_1, 0, -\frac{2(n+1)b_n}{n} E_1 - 2b_n \sum_{\beta=2}^{\alpha-1} \frac{c_{n,\beta}}{n+1-\beta} E_\beta + 2b_n c_{n,\alpha} E_\alpha, 0, 0 \right), \\ F_{n+1n+1} = F_{2n+1 2n+1} &= \left(-a_n E_1, 0, -\frac{2(n+1)b_n}{n} E_1 - 2b_n \sum_{\beta=2}^{n-1} \frac{c_{n,\beta}}{n+1-\beta} E_\beta - 2b_n c_{n,n} E_n, 0, 0 \right), \end{aligned}$$

$$F_{1k} = (0, a_n E_k, 0, 0, 0), \quad F_{1n+k} = (-a_n E_k, 0, 0, 0, 0), \quad F_{kn+k} = (0, a_n E_1, 0, 0, 0),$$

$$F_{kl} = F_{n+k n+l} = (0, 0, 0, b_n d_n E_{kl}, 0), \quad F_{kn+l} = -F_{ln+k} = (0, 0, 0, 0, -b_n d_n E_{kl}),$$

where $2 \leq \alpha \leq n$, $2 \leq k \leq n+1$ and $2 \leq k < l \leq n+1$. Define

$$H_{AB} = \begin{cases} \frac{1}{a_n^2} F_{11}, & A = 1, B = 1 \\ \frac{1}{a_n \sqrt{a_n^2 + b_n^2 d_n^2}} F_{1B}, & A = 1, B \neq 1, \\ \frac{1}{a_n^2 + b_n^2 d_n^2} F_{AB}, & A \neq 1, B \neq 1. \end{cases}$$

Then, at P , the second fundamental form of F is given in terms of the frame $\{e_A, e_\tau, e_{kl}, e_{\bar{k}\bar{l}} \mid 1 \leq A \leq 2n+1, 2n+2 \leq \tau \leq 5n+1, 2 \leq k < l \leq n+1\}$ by

$$h_{AB}^\tau = \langle H_{AB}, e_\tau \rangle, \quad h_{AB}^{(k,l)} = \langle H_{AB}, e_{kl} \rangle, \quad h_{AB}^{(\bar{k},\bar{l})} = \langle H_{AB}, e_{\bar{k}\bar{l}} \rangle. \quad (3.5)$$

More explicitly, we gain

Proposition 3.2. *The second fundamental form of F at the base point P w.r.t. the frame $\{e_A, e_\tau, e_{kl}, e_{\bar{k}\bar{l}} \mid 1 \leq A \leq 2n+1, 2n+2 \leq \tau \leq 5n+1, 2 \leq k < l \leq n+1\}$ is given by*

- (1) $h_{11}^{2n+2} = \frac{b_n}{a_n}, \quad h_{AA}^{2n+2} = -\frac{(n+2)a_n b_n}{n(a_n^2 + b_n^2 d_n^2)}, \quad 2 \leq A \leq 2n+1;$
- (2) $h_{1n+k}^{2n+1+k} = h_{1k}^{3n+1+k} = \frac{b_n d_n}{a_n^2 + b_n^2 d_n^2}, \quad 2 \leq k \leq n+1;$
- (3) $h_{kk}^{4n+1+\alpha} = h_{n+kn+k}^{4n+1+\alpha} = -\frac{2b_n c_{n,\alpha}}{(n+1-\alpha)(a_n^2 + b_n^2 d_n^2)}, \quad 2 \leq \alpha < n, \quad \alpha < k \leq n+1;$
- (4) $h_{\alpha\alpha}^{4n+1+\alpha} = h_{n+\alpha n+\alpha}^{4n+1+\alpha} = \frac{2b_n c_{n,\alpha}}{a_n^2 + b_n^2 d_n^2}, \quad 2 \leq \alpha \leq n;$
- (5) $h_{n+1n+1}^{5n+1} = h_{2n+12n+1}^{5n+1} = -\frac{2b_n c_{n,n}}{a_n^2 + b_n^2 d_n^2};$
- (6) $h_{kl}^{(k,l)} = h_{n+kn+l}^{(k,l)} = h_{kn+l}^{(\bar{k},\bar{l})} = -h_{ln+k}^{(\bar{k},\bar{l})} = \frac{b_n d_n}{a_n^2 + b_n^2 d_n^2}, \quad 2 \leq k < l \leq n+1;$

with the same value in the symmetric slot and zero for others.

Proof. It follows by direct computation. Moreover, one can easily see that F is minimal. \square

3.2 Type-II minimal spheres

In terms of quaternions,

$$S^{4n+3}(1) = \left\{ (a_1, \dots, a_{n+1}) \in \mathbf{H}^{n+1} \mid \sum_{k=1}^{n+1} |a_k|^2 = 1 \right\}.$$

A function in $H_{1,1}^{n+1} \cap C^\infty(\mathbf{H}^{n+1}, \mathbf{R})$ restricted to S^{4n+3} is S^3 -invariant. As a consequence, it descends to a function on \mathbf{HP}^n . Let $\phi \in H_{1,1}^{n+1} \cap C^\infty(\mathbf{H}^{n+1}, \mathbf{R})$. Then $\phi = \sum_{k,l=1}^{n+1} \lambda_{\bar{k}l} \bar{a}_k a_l$. Notice that $D = -4 \sum_{k=1}^{n+1} \frac{\partial^2}{\partial a_k \partial \bar{a}_k}$ in terms of quaternions, so conditions $D\phi = 0$ and $\bar{\phi} = \phi$ are equivalent to

$$\sum_{k=1}^{n+1} \lambda_{\bar{k}k} = 0, \quad \lambda_{\bar{k}l} = \lambda_{l\bar{k}},$$

where $\lambda_{k\bar{l}} = \bar{\lambda}_{\bar{k}l}$. Similarly, we have

Lemma 3.3. *There is an orthonormal basis of $H_{1,1}^{n+1} \cap C^\infty(\mathbf{H}^{n+1}, \mathbf{R})$ w.r.t. the L^2 -inner product given by*

$$\phi_\alpha = c_{n,\alpha} \left(|z_{2\alpha-1}|^2 + |z_{2\alpha}|^2 - \frac{1}{n+1-\alpha} \sum_{k=\alpha+1}^{n+1} (|z_{2k-1}|^2 + |z_{2k}|^2) \right), \quad 1 \leq \alpha \leq n,$$

$$\phi_{kl} = d_n \operatorname{Re}(\bar{z}_{2k-1} \bar{z}_{2l-1} + z_{2k} \bar{z}_{2l}), \quad \phi_{\bar{k}\bar{l}} = d_n \operatorname{Im}(\bar{z}_{2k-1} \bar{z}_{2l-1} + z_{2k} \bar{z}_{2l}), \quad 1 \leq k < l \leq n+1,$$

$$\tilde{\phi}_{kl} = d_n \operatorname{Re}(\bar{z}_{2k-1} \bar{z}_{2l} - z_{2k} \bar{z}_{2l-1}), \quad \tilde{\phi}_{\bar{k}\bar{l}} = d_n \operatorname{Im}(\bar{z}_{2k-1} \bar{z}_{2l} - z_{2k} \bar{z}_{2l-1}), \quad 1 \leq k < l \leq n+1,$$

where $c_{n,\alpha} = \sqrt{\frac{(n+1)(n+1-\alpha)}{n(n+2-\alpha)}}$, $d_n = \sqrt{\frac{2(n+1)}{n}}$ and we consider a_k as $z_{2k-1} + z_{2k}j$.

Proof. Similar to the proof of Lemma 3.1. □

Thus, we have an isometric minimal immersion $\Phi : \mathbf{HP}^n \longrightarrow S^{2n^2+7n+3}(1)$ by $[a] \mapsto (\phi_\alpha(a), \phi_{kl}(a), \phi_{\bar{k}\bar{l}}(a), \tilde{\phi}_{kl}(a), \tilde{\phi}_{\bar{k}\bar{l}}(a))$, and a Lawson-Osserman cone determined by $f' = \Phi \circ \pi^{4n+3, 4n}$. It is known that f' is just the Hopf map η' when $n = 1$. The type-II Lawson-Osserman minimal sphere is represented by $F' : S^{4n+3} \longrightarrow S^{2n^2+7n+3}$ sending

$$a \mapsto (\tilde{a}_n(\operatorname{Re}(z), \operatorname{Im}(z)), \tilde{b}_n(\phi_\alpha(a), \phi_{kl}(a), \phi_{\bar{k}\bar{l}}(a), \tilde{\phi}_{kl}(a), \tilde{\phi}_{\bar{k}\bar{l}}(a))),$$

where $\tilde{a}_n = \sqrt{\frac{6(n+1)}{(n+2)(4n+3)}}$ and $\tilde{b}_n = \sqrt{\frac{n(4n+5)}{(n+2)(4n+3)}}$. For the choice of \tilde{a}_n and \tilde{b}_n we refer to [XYZ] for a general explanation. By the construction of Φ , F is homogeneous and its image is an $\operatorname{Id} \oplus \rho_1(Sp(n+1))$ -orbit through the base point

$$P = (a_n E_1, 0, b_n E_1, 0, 0, 0, 0),$$

where ρ_1 is defined in (3.1). Next, we compute its second fundamental form at P .

Using (2.1) and (2.2) with $m = 2n + 1$, we have

$$\begin{aligned}
\phi_1 &= 1 - \frac{n+1}{n} \sum_{k=2}^{n+1} \left(t_{2k-1}^2 + t_{2k}^2 + t_{2n+2k}^2 + t_{2n+1+2k}^2 \right) + o(t^2), \\
\phi_\alpha &= c_{n,\alpha} \left[t_{2\alpha-1}^2 + t_{2\alpha}^2 + t_{2n+2\alpha}^2 + t_{2n+1+2\alpha}^2 - \frac{1}{n+1-\alpha} \sum_{k=\alpha+1}^{n+1} (t_{2k-1}^2 + t_{2k}^2 \right. \\
&\quad \left. + t_{2n+2k}^2 + t_{2n+1+2k}^2) \right] + o(t^2), \\
\phi_{1k} &= d_n \left(t_{2k-1} + t_2 t_{2k} + t_{2n+3} t_{2n+1+2k} \right) + o(t^2), \\
\phi_{1\bar{k}} &= d_n \left(t_{2n+2k} - t_2 t_{2n+1+2k} + t_{2k} t_{2n+3} \right) + o(t^2), \\
\phi_{kl} &= d_n \left(t_{2k-1} t_{2l-1} + t_{2k} t_{2l} + t_{2n+2k} t_{2n+2l} + t_{2n+1+2k} t_{2n+1+2l} \right) + o(t^2) \quad (3.6) \\
\phi_{\bar{k}\bar{l}} &= d_n \left(t_{2k-1} t_{2n+2l} - t_{2k} t_{2n+1+2l} - t_{2l-1} t_{2n+2k} + t_{2l} t_{2n+1+2k} \right) + o(t^2), \\
\tilde{\phi}_{1k} &= d_n \left(t_{2k} - t_2 t_{2k-1} - t_{2n+3} t_{2n+2k} \right) + o(t^2), \\
\tilde{\phi}_{1\bar{k}} &= d_n \left(t_{2n+1+2k} + t_2 t_{2n+2k} - t_{2k-1} t_{2n+3} \right) + o(t^2), \\
\tilde{\phi}_{kl} &= d_n \left(t_{2k-1} t_{2l} - t_{2k} t_{2l-1} + t_{2n+2k} t_{2n+1+2l} - t_{2n+1+2k} t_{2n+2l} \right) + o(t^2), \\
\tilde{\phi}_{\bar{k}\bar{l}} &= d_n \left(t_{2k-1} t_{2n+1+2l} - t_{2k} t_{2n+2l} - t_{2l-1} t_{2n+1+2k} - t_{2l} t_{2n+2k} \right) + o(t^2),
\end{aligned}$$

where $2 \leq \alpha \leq n$, $2 \leq k \leq n+1$ and $2 \leq k < l \leq n+1$. Noticing that $F'_*(\epsilon_A) = \frac{\partial F'}{\partial t_A} \Big|_{t=0}$, by (3.6), (2.1) and (2.2), we get

$$\begin{aligned}
F'_*(\epsilon_1) &= (0, \tilde{a}_n E_1, 0, 0, 0, 0, 0), \\
F'_*(\epsilon_2) &= (\tilde{a}_n E_2, 0, 0, 0, 0, 0, 0) \\
F'_*(\epsilon_{2n+3}) &= (0, \tilde{a}_n E_2, 0, 0, 0, 0, 0), \\
F'_*(\epsilon_{2k-1}) &= (\tilde{a}_n E_{2k-1}, 0, 0, \tilde{b}_n d_n E_{1k}, 0, 0, 0), \\
F'_*(\epsilon_{2k}) &= (\tilde{a}_n E_{2k}, 0, 0, 0, 0, \tilde{b}_n d_n E_{1k}, 0), \\
F'_*(\epsilon_{2n+2k}) &= (0, \tilde{a}_n E_{2k-1}, 0, 0, \tilde{b}_n d_n E_{1k}, 0, 0), \\
F'_*(\epsilon_{2n+1+2k}) &= (0, \tilde{a}_n E_{2k}, 0, 0, 0, 0, \tilde{b}_n d_n E_{1k}),
\end{aligned}$$

where $2 \leq k \leq n+1$. Further, we obtain an orthonormal basis of $F'_*(T_p S^{4n+3})$:

$$e_A = \begin{cases} \frac{1}{\tilde{a}_n} F'_*(\epsilon_A), & A = 1, 2, 2n+3, \\ \frac{1}{\sqrt{\tilde{a}_n^2 + \tilde{b}_n^2 d_n^2}} F'_*(\epsilon_A), & A \neq 1, 2, 2n+3. \end{cases}$$

and an orthonormal basis for the normal space of $F'_*(T_p S^{4n+3})$ in $T_p S^{2n^2+7n+3}$:

$$\begin{aligned}
e_{4n+4} &= (-\tilde{b}_n E_1, 0, \tilde{a}_n E_1, 0, 0, 0), \\
e_{4n+2k+1} &= \frac{1}{\sqrt{\tilde{a}_n^2 + \tilde{b}_n^2 d_n^2}} (-\tilde{b}_n d_n E_{2k-1}, 0, 0, \tilde{a}_n E_{1k}, 0, 0, 0), \\
e_{4n+2k+2} &= \frac{1}{\sqrt{\tilde{a}_n^2 + \tilde{b}_n^2 d_n^2}} (-\tilde{b}_n d_n E_{2k}, 0, 0, 0, 0, \tilde{a}_n E_{1k}, 0), \\
e_{6n+2k+1} &= \frac{1}{\sqrt{\tilde{a}_n^2 + \tilde{b}_n^2 d_n^2}} (0, -\tilde{b}_n d_n E_{2k-1}, 0, 0, \tilde{a}_n E_{1k}, 0, 0), \\
e_{6n+2k+2} &= \frac{1}{\sqrt{\tilde{a}_n^2 + \tilde{b}_n^2 d_n^2}} (0, -\tilde{b}_n d_n E_{2k}, 0, 0, 0, 0, \tilde{a}_n E_{1k}), \\
e_{8n+3+\alpha} &= (0, 0, E_\alpha, 0, 0, 0, 0),
\end{aligned}$$

and

$$\begin{aligned}
e_{kl} &= (0, 0, 0, E_{kl}, 0, 0, 0), & e_{\bar{k}\bar{l}} &= (0, 0, 0, 0, E_{kl}, 0, 0), \\
\tilde{e}_{kl} &= (0, 0, 0, 0, 0, E_{kl}, 0), & \tilde{e}_{\bar{k}\bar{l}} &= (0, 0, 0, 0, 0, 0, E_{kl}),
\end{aligned}$$

where $2 \leq k \leq n+1$, $2 \leq \alpha \leq n$ and $2 \leq k < l \leq n+1$.

We set $F'_{AB} = \frac{\partial^2 F'}{\partial t_A \partial t_B} \Big|_{t=0}$ and define

$$H_{AB} = \begin{cases} \frac{1}{\tilde{a}_n^2} F'_{AB}, & A = 1, 2, 2n+3, B = 1, 2, 2n+3 \\ \frac{1}{\tilde{a}_n \sqrt{\tilde{a}_n^2 + \tilde{b}_n^2 d_n^2}} F'_{AB}, & A = 1, 2, 2n+3, B \neq 1, 2, 2n+3, \\ \frac{1}{\tilde{a}_n^2 + \tilde{b}_n^2 d_n^2} F'_{AB}, & A, B \neq 1, 2, 2n+3. \end{cases}$$

Then, at P , the second fundamental form of F' is given in terms of the frame $\{e_A, e_\tau, e_{kl}, e_{\bar{k}\bar{l}}, \tilde{e}_{kl}, \tilde{e}_{\bar{k}\bar{l}} \mid 1 \leq A \leq 4n+3, 4n+4 \leq \tau \leq 9n+3, 2 \leq k < l \leq n+1\}$ by

$$\begin{aligned}
h_{AB}^\tau &= \langle H_{AB}, e_\tau \rangle, & h_{AB}^{(k,l)} &= \langle H_{AB}, e_{kl} \rangle, & h_{AB}^{(\bar{k},\bar{l})} &= \langle H_{AB}, e_{\bar{k}\bar{l}} \rangle, \\
h_{AB}^{[k,l]} &= \langle H_{AB}, \tilde{e}_{kl} \rangle, & h_{AB}^{[\bar{k},\bar{l}]} &= \langle H_{AB}, \tilde{e}_{\bar{k}\bar{l}} \rangle.
\end{aligned}$$

In summary, we have

Proposition 3.4. *The second fundamental form of F' at the base point P w.r.t. the frame $\{e_A, e_\tau, e_{kl}, e_{\bar{k}\bar{l}}, \tilde{e}_{kl}, \tilde{e}_{\bar{k}\bar{l}} \mid 1 \leq A \leq 4n+3, 4n+4 \leq \tau \leq 9n+3, 2 \leq k < l \leq n+1\}$ is given by*

$$\begin{aligned}
(1) \quad & h_{AA}^{4n+4} = \frac{\tilde{b}_n}{\tilde{a}_n}, \quad h_{BB}^{4n+4} = -\frac{(n+2)\tilde{a}_n\tilde{b}_n}{n(\tilde{a}_n^2+\tilde{b}_n^2d_n^2)}, \quad A = 1, 2, 2n+3, B \neq 1, 2, 2n+3; \\
(2) \quad & h_{1\ 2n+2k}^{4n+1+2k} = h_{2\ 2k}^{4n+1+2k} = h_{2n+3\ 2n+1+2k}^{4n+1+2k} = \frac{\tilde{b}_nd_n}{\tilde{a}_n^2+\tilde{b}_n^2d_n^2}, \quad 2 \leq k \leq n+1; \\
(3) \quad & h_{1\ 2n+1+2k}^{4n+2+2k} = -h_{2\ 2k-1}^{4n+2+2k} = -h_{2n+3\ 2n+2k}^{4n+2+2k} = \frac{\tilde{b}_nd_n}{\tilde{a}_n^2+\tilde{b}_n^2d_n^2}, \quad 2 \leq k \leq n+1; \\
(4) \quad & h_{1\ 2k-1}^{6n+1+2k} = h_{2\ 2n+1+2k}^{6n+1+2k} = -h_{2n+3\ 2k}^{6n+1+2k} = -\frac{\tilde{b}_nd_n}{\tilde{a}_n^2+\tilde{b}_n^2d_n^2}, \quad 2 \leq k \leq n+1; \\
(5) \quad & h_{1\ 2k}^{6n+2+2k} = -h_{2\ 2n+2k}^{6n+2+2k} = h_{2n+3\ 2k-1}^{6n+2+2k} = -\frac{\tilde{b}_nd_n}{\tilde{a}_n^2+\tilde{b}_n^2d_n^2}, \quad 2 \leq k \leq n+1; \\
(6) \quad & h_{2k-1\ 2k-1}^{8n+3+\alpha} = h_{2k\ 2k}^{8n+3+\alpha} = h_{2n+2k\ 2n+2k}^{8n+3+\alpha} = h_{2n+1+2k\ 2n+1+2k}^{8n+3+\alpha} = -\frac{2\tilde{b}_nc_{n,\alpha}}{(n+1-\alpha)(\tilde{a}_n^2+\tilde{b}_n^2d_n^2)}, \\
& 2 \leq \alpha < n, \alpha < k \leq n+1; \\
(7) \quad & h_{2\alpha-1\ 2\alpha-1}^{8n+3+\alpha} = h_{2\alpha\ 2\alpha}^{8n+3+\alpha} = h_{2n+2\alpha\ 2n+2\alpha}^{8n+3+\alpha} = h_{2n+1+2\alpha\ 2n+1+2\alpha}^{8n+3+\alpha} = \frac{2\tilde{b}_nc_{n,\alpha}}{\tilde{a}_n^2+\tilde{b}_n^2d_n^2}, \quad 2 \leq \alpha \leq n; \\
(8) \quad & h_{2n+1\ 2n+1}^{9n+3} = h_{2n+2\ 2n+2}^{9n+3} = h_{4n+2\ 4n+2}^{9n+3} = h_{4n+3\ 4n+3}^{9n+3} = -\frac{2\tilde{b}_nc_{n,n}}{\tilde{a}_n^2+\tilde{b}_n^2d_n^2}; \\
(9) \quad & h_{2k-1\ 2l-1}^{(k,l)} = h_{2k\ 2l}^{(k,l)} = h_{2n+2k\ 2n+2l}^{(k,l)} = h_{2n+1+2k\ 2n+1+2l}^{(k,l)} = h_{2k-1\ 2n+2l}^{(\bar{k},\bar{l})} = -h_{2k\ 2n+1+2l}^{(\bar{k},\bar{l})} = \\
& -h_{2l-1\ 2n+2k}^{(\bar{k},\bar{l})} = h_{2l\ 2n+1+2k}^{(\bar{k},\bar{l})} = \frac{\tilde{b}_nd_n}{\tilde{a}_n^2+\tilde{b}_n^2d_n^2}, \quad 2 \leq k < l \leq n+1; \\
(10) \quad & h_{2k-1\ 2l}^{[k,l]} = -h_{2k\ 2l-1}^{[k,l]} = h_{2n+2k\ 2n+1+2l}^{[k,l]} = -h_{2n+1+2k\ 2n+2l}^{[k,l]} = h_{2k-1\ 2n+1+2l}^{[\bar{k},\bar{l}]} = h_{2k\ 2n+2l}^{[\bar{k},\bar{l}]} = \\
& -h_{2l-1\ 2n+1+2k}^{[\bar{k},\bar{l}]} = -h_{2l\ 2n+2k}^{[\bar{k},\bar{l}]} = \frac{\tilde{b}_nd_n}{\tilde{a}_n^2+\tilde{b}_n^2d_n^2}, \quad 2 \leq k < l \leq n+1;
\end{aligned}$$

with the same value in the symmetric slot and zero for others.

Proof. By direct computation. Moreover, one can see that F' is minimal. \square

3.3 Type-III minimal spheres

Let $p_1, p_2 \in \mathbf{O}$ written as $p_1 = (z_1 + z_2j) + (z_3 + z_4j)e$ and $p_2 = (z_5 + z_6j) + (z_7 + z_8j)e$. Then,

$$\begin{aligned}
|p_1|^2 - |p_2|^2 &= \sum_{k=1}^4 |z_k|^2 - \sum_{k=5}^8 |z_k|^2, \\
2\bar{p}_1 p_2 &= (a_1 + a_2j) + (a_3 + a_4j)e,
\end{aligned}$$

where

$$\begin{aligned}
a_1 &= \bar{z}_1 z_5 + z_2 \bar{z}_6 + z_3 \bar{z}_7 + \bar{z}_4 z_8, & a_2 &= \bar{z}_1 z_6 - z_2 \bar{z}_5 - \bar{z}_3 z_8 + z_4 \bar{z}_7, \\
a_3 &= \bar{z}_1 z_7 + \bar{z}_2 z_8 - z_3 \bar{z}_5 - z_4 \bar{z}_6, & a_4 &= z_1 z_8 - z_2 z_7 + z_3 z_6 - z_4 z_5.
\end{aligned}$$

Set

$$f_1 = |p_1|^2 - |p_2|^2, \quad f_{1+k} = 2\operatorname{Re}(a_k), \quad f_{5+k} = 2\operatorname{Im}(a_k), \quad 1 \leq k \leq 4,$$

Then the third Hopf map $\eta'' : S^{15} \longrightarrow S^8$ is given by $z \mapsto (f_1, \dots, f_9)$, where $p = (p_1, p_2)$ is identified with $z = (z_1, \dots, z_8)$. The Type-III Lawson-Osserman minimal sphere is represented by

$$F'' : S^{15}(1) \longrightarrow S^{24}(1), \quad z \mapsto \left(\sqrt{\frac{28}{45}}(\operatorname{Re}(z), \operatorname{Im}(z)), \sqrt{\frac{17}{45}}(f_1, f_2, \dots, f_9) \right).$$

It is known that F'' is also homogeneous.

We will compute the second fundamental form of F'' at point $P = \left(\sqrt{\frac{28}{45}}E_1, 0, \sqrt{\frac{17}{45}}E_1\right)$. Substituting (2.1) and (2.2) with $m = 7$ into f_k , we obtain

$$\begin{aligned}
f_1 &= 1 - 2 \sum_{k=5}^8 (t_k^2 + t_{7+k}^2) + o(t^2), \\
f_2 &= 2(t_5 + t_2t_6 + t_3t_7 + t_4t_8 + t_9t_{13} + t_{10}t_{14} + t_{11}t_{15}) + o(t^2), \\
f_3 &= 2(t_6 - t_2t_5 - t_3t_8 + t_4t_7 - t_9t_{12} - t_{10}t_{15} + t_{11}t_{14}) + o(t^2), \\
f_4 &= 2(t_7 + t_2t_8 - t_3t_5 - t_4t_6 + t_9t_{15} - t_{10}t_{12} - t_{11}t_{13}) + o(t^2), \\
f_5 &= 2(t_8 - 2t_1t_{15} - t_2t_7 + t_3t_6 - t_4t_5 + t_9t_{14} - t_{10}t_{13} + t_{11}t_{12}) + o(t^2), \\
f_6 &= 2(t_{12} - t_2t_{13} - t_3t_{14} + t_4t_{15} + t_6t_9 + t_7t_{10} - t_8t_{11}) + o(t^2), \\
f_7 &= 2(t_{13} + t_2t_{12} - t_3t_{15} - t_4t_{14} - t_5t_9 + t_7t_{11} + t_8t_{10}) + o(t^2), \\
f_8 &= 2(t_{14} + t_2t_{15} + t_3t_{12} + t_4t_{13} - t_5t_{10} - t_6t_{11} - t_8t_9) + o(t^2), \\
f_9 &= 2(t_{15} + 2t_1t_8 - t_2t_{14} + t_3t_{13} - t_4t_{12} - t_5t_{11} + t_6t_{10} - t_7t_9) + o(t^2).
\end{aligned}$$

Taking the partial derivative w.r.t. t_A , at $t = 0$, we have

$$\begin{aligned}
F''_*(\epsilon_1) &= \left(0, \sqrt{\frac{28}{45}}E_1, 0\right), \quad F''_*(\epsilon_k) = \left(\sqrt{\frac{28}{45}}E_k, 0, 0\right), \quad F''_*(\epsilon_l) = \left(\sqrt{\frac{28}{45}}E_l, 0, \sqrt{\frac{17}{45}}E_{l-3}\right), \\
F''_*(\epsilon_{7+k}) &= \left(0, \sqrt{\frac{28}{45}}E_k, 0\right), \quad F''_*(\epsilon_{7+l}) = \left(0, \sqrt{\frac{28}{45}}E_l, \sqrt{\frac{17}{45}}E_{l+1}\right),
\end{aligned}$$

where $2 \leq k \leq 3$ and $5 \leq l \leq 8$. Further, we gain an orthonormal basis of $F''_*(T_p S^{15})$:

$$e_A = \begin{cases} \sqrt{\frac{45}{28}}F''_*(\epsilon_A), & A = 1, 2, 3, 4, 9, 10, 11, \\ \sqrt{\frac{15}{32}}F''_*(\epsilon_A), & A = 5, 6, 7, 8, 12, 13, 14, 15. \end{cases}$$

and an orthonormal basis for the normal space of $F''_*(T_p S^{15})$ in $T_p S^{24}$:

$$\begin{aligned}
e_{16} &= \left(-\sqrt{\frac{17}{45}}E_1, 0, \sqrt{\frac{28}{45}}E_1\right), \\
e_{12+k} &= \left(-\sqrt{\frac{17}{24}}E_k, 0, \sqrt{\frac{7}{24}}E_{k-3}\right), \\
e_{16+k} &= \left(0, -\sqrt{\frac{17}{24}}E_l, \sqrt{\frac{7}{24}}E_{l+1}\right).
\end{aligned}$$

where $5 \leq k \leq 8$.

We set $F''_{AB} = \frac{\partial^2 F''}{\partial t_A \partial t_B} \Big|_{t=0}$ and define

$$H_{AB} = \begin{cases} \frac{45}{28} F''_{AB}, & A, B = 1, 2, 3, 4, 9, 10, 11, \\ \sqrt{\frac{3}{14}} \cdot \frac{15}{8} F''_{AB}, & A = 1, 2, 3, 4, 9, 10, 11, B = 5, 6, 7, 8, 12, 13, 14, 15, \\ \frac{15}{32} F''_{AB}, & A, B = 5, 6, 7, 8, 12, 13, 14, 15. \end{cases}$$

Then, at P , the second fundamental form of F'' is given in terms of the frame $\{e_A, e_\tau, | 1 \leq A \leq 15, 16 \leq \tau \leq 24\}$ by $h_{AB}^\tau = \langle H_{AB}, e_\tau \rangle$.

In summary, we have

Proposition 3.5. *The second fundamental form of F'' at the base point P w.r.t. the frame $\{e_A, e_\tau | 1 \leq A \leq 15, 16 \leq \tau \leq 24\}$ is given by*

- (1) $h_{11}^{16} = h_{kk}^{16} = h_{7+k, 7+k}^{16} = \sqrt{\frac{17}{28}}, h_{ll}^{16} = h_{l+7, l+7}^{16} = -\frac{\sqrt{119}}{16}, 2 \leq k \leq 4, 5 \leq l \leq 8;$
- (2) $h_{12}^{17} = h_{26}^{17} = h_{37}^{17} = h_{48}^{17} = h_{913}^{17} = h_{1014}^{17} = h_{1115}^{17} = \frac{\sqrt{85}}{16};$
- (3) $h_{13}^{18} = -h_{25}^{18} = -h_{38}^{18} = h_{47}^{18} = -h_{912}^{18} = -h_{1015}^{18} = h_{1114}^{18} = \frac{\sqrt{85}}{16};$
- (4) $h_{14}^{19} = h_{28}^{19} = -h_{35}^{19} = -h_{46}^{19} = h_{915}^{19} = -h_{1012}^{19} = -h_{1113}^{19} = \frac{\sqrt{85}}{16};$
- (5) $h_{15}^{20} = h_{27}^{20} = -h_{36}^{20} = h_{45}^{20} = -h_{914}^{20} = h_{1013}^{20} = -h_{1112}^{20} = -\frac{\sqrt{85}}{16};$
- (6) $h_{15}^{21} = h_{213}^{21} = h_{314}^{21} = -h_{415}^{21} = -h_{69}^{21} = -h_{710}^{21} = h_{811}^{21} = -\frac{\sqrt{85}}{16};$
- (7) $h_{16}^{22} = -h_{212}^{22} = h_{315}^{22} = h_{414}^{22} = h_{59}^{22} = -h_{711}^{22} = -h_{810}^{22} = -\frac{\sqrt{85}}{16};$
- (8) $h_{17}^{23} = -h_{215}^{23} = -h_{312}^{23} = -h_{413}^{23} = h_{510}^{23} = h_{611}^{23} = h_{89}^{23} = -\frac{\sqrt{85}}{16};$
- (9) $h_{18}^{24} = h_{214}^{24} = h_{313}^{24} = -h_{412}^{24} = -h_{511}^{24} = h_{610}^{24} = -h_{79}^{24} = \frac{\sqrt{85}}{16};$

with the same value in the symmetric slot and zero for others.

Proof. By computation. Moreover, one can see that F'' is minimal. □

4 On the area-minimizing property

4.1 Lawlor's criterion

For completeness, we briefly recall Lawlor's curvature criterion for proving a minimal cone to be area-minimizing. For further details readers are referred to [Law91].

Let Σ be a smooth n dimensional submanifold of unit sphere S^N and

$$\mathcal{C}\Sigma = \{tx : t \in [0, \infty) \text{ and } x \in \Sigma\}.$$

Fix $p \in \Sigma$. A *normal geodesic* of length ℓ is an arc of a great circle γ which is perpendicular to Σ at its starting point $\gamma(0) = p$. We call γ an *open normal geodesic* if we leave off the endpoint $\gamma(\ell)$. Let $U_p(\ell)$ be the union of open normal geodesics from p of length ℓ . Then *normal wedge* $W_p(\ell)$ is defined to be $\mathcal{C}U_p(\ell) - \{0\}$. The *normal radius* of $\mathcal{C}\Sigma$ at a point $p \in \Sigma$ is the largest ℓ_p such that $W_p(\ell_p)$ intersects $\mathcal{C}\Sigma$ only in the ray \vec{op} .

Suppose $p \in \Sigma$ and v is a unit vector in the normal space $T_p^\perp \Sigma$. Let (r, θ) be the polar coordinate of the plane spanned by \vec{op} and v . A *projection curve* γ_p , if exists, satisfies

$$(ODE) \quad \begin{cases} \frac{dr}{d\theta} = r \sqrt{r^{2n+2} \cos^{2n} \theta \inf_{v \in T_p^\perp \Sigma, |v|=1} \left(\det(I - \tan \theta (h_{AB}^v)) \right)^2 - 1}, \\ r(0) = 1, \end{cases}$$

where (h_{AB}^v) is the matrix of the second fundamental form of Σ at p , in the normal direction v . The existence of the ODE relies on the size of second fundamental form and $\dim \mathcal{C}\Sigma$. If γ_p exists, either $\frac{dr}{d\theta}$ vanishes at some positive $\theta(p)$, or r goes to infinity as θ approaches some finite value $\theta_0(p)$. In the latter case, we call the smallest $\theta_0(p)$ the vanishing angle at p . Let Γ_p be rotated surface generated by γ_p in $W_p(\theta_0(p))$. Then we define Π_p by sending Γ_p to p and requiring $\Pi_p(tz) = t\Pi_p(z)$ for $t > 0$ and $z \in \Gamma_p$. If $\{W_p(\theta_0(p)) : p \in \Sigma\}$ do not intersect, we assemble $\{\Pi_p : p \in \Sigma\}$ together and extend it to a global *retraction* $\Pi : \mathbf{R}^{N+1} \rightarrow \mathcal{C}\Sigma$ which equals Π_p in $W_p(\theta_0(p))$ and collapses everything else to 0. It can be guaranteed by (ODE) that Π is a continuously area-noincreasing projection to $\mathcal{C}\Sigma$.

By using the retraction Π , Lawlor proved

Theorem 4.1. (Lawlor's curvature criterion [Law91]) *Let Σ be a smooth n dimensional submanifold of unit sphere S^N . Suppose $\ell_0 = \min_{p \in \Sigma} \ell_p > 2 \max_{p \in \Sigma} \theta_0(p)$ which ensures that $\{W_p(\theta_0(p)) : p \in \Sigma\}$ do not intersect. Then $\mathcal{C}\Sigma$ is area-minimizing (in the sense of mod 2 when Σ is unorientable).*

Remark. Lawlor made a table (page 20-21 in [Law91]) of estimated vanishing angles for $\dim \mathcal{C}\Sigma \leq 12$ and \mathcal{S}^2 where

$$\mathcal{S} = \max_{p \in \Sigma} \left(\sup_{v \in T_p^\perp \Sigma, |v|=1} \left(\sum_{A,B} (h_{AB}^v)^2 \right)^{\frac{1}{2}} \right). \quad (4.1)$$

He used the control

$$\inf_{v \in T_p^\perp \Sigma, |v|=1} \left(\det(I - t (h_{AB}^v)) \right) \geq (1 - \mathcal{S}t)e^{\mathcal{S}t} \quad (4.2)$$

for $\dim \mathcal{C}\Sigma = 12$ and a more accurate lower bound $F(\mathcal{S}, t, \dim \mathcal{C}\Sigma)$ for $\dim \mathcal{C}\Sigma < 12$. By $V(m, \mathcal{S})$ we mean the estimated vanishing angle based on (4.2) for $\dim \mathcal{C}\Sigma \geq 12$ and \mathcal{S} . When $m = \dim \mathcal{C}\Sigma > 12$, Lawlor proved the following nice property

$$\tan \left(V(m, \frac{m}{12} \mathcal{S}) \right) < \frac{12}{m} \tan \left(V(12, \mathcal{S}) \right). \quad (4.3)$$

Moreover, we remark that

$$V(m, a) < V(m, b) \text{ for } a < b. \quad (4.4)$$

4.2 Proof for the area-minimizing property

Let $\mathcal{C}F$, $\mathcal{C}F'$ and $\mathcal{C}F''$ denote the cones over images of F , F' and F'' respectively. Then, we have

Theorem 4.2. *The minimal cones $\mathcal{C}F$, $\mathcal{C}F'$ and $\mathcal{C}F''$ are area-minimizing.*

Proof. To reduce redundancy, we present a complete proof only for $\mathcal{C}F$. We will show

- (1) Any normal line through P intersects $\mathcal{C}F$ only at P , i.e., the normal radius $\ell_0 \geq \frac{\pi}{2}$;
- (2) The vanishing angle $\theta_0 < \frac{\pi}{4}$;

and the theorem follows from the Lawlor's criterion.

Since F , F' and F'' are homogeneous, it is sufficient to do calculations at the base point P . We verify (1) first. Let X be normal vector through P . Then, according to (3.4), X can be written as

$$\begin{aligned} X = P &+ \lambda_{2n+2} e_{2n+2} + \sum_{k=2}^{n+1} \left(\lambda_{2n+1+k} e_{2n+1+k} + \lambda_{3n+1+k} e_{3n+1+k} \right) \\ &+ \sum_{\alpha=2}^n \lambda_{4n+1+\alpha} e_{4n+1+\alpha} + \sum_{2 \leq k < l \leq n+1} \left(\lambda_{kl} e_{kl} + \lambda_{\bar{k}\bar{l}} e_{\bar{k}\bar{l}} \right). \end{aligned}$$

In terms of blocks, we write $X = (\xi, \eta, \mu, \varsigma, \tau)$ and (3.4) gives

$$\begin{aligned} \xi_1 &= a_n - b_n \lambda_{2n+2}, & \xi_k &= -\frac{b_n d_n}{\sqrt{a_n^2 + b_n^2 d_n^2}} \lambda_{2n+1+k}, \quad 2 \leq k \leq n+1, \\ \eta_1 &= 0, & \eta_k &= \frac{b_n d_n}{\sqrt{a_n^2 + b_n^2 d_n^2}} \lambda_{3n+1+k}, \quad 2 \leq k \leq n+1, \\ \mu_1 &= b_n + a_n \lambda_{2n+2}, & \mu_\alpha &= \lambda_{4n+1+\alpha}, \quad 2 \leq \alpha \leq n, \\ \varsigma_{1l} &= \frac{a_n}{\sqrt{a_n^2 + b_n^2 d_n^2}} \lambda_{2n+1+l}, \quad 2 \leq l \leq n+1, & \varsigma_{kl} &= \lambda_{kl}, \quad 2 \leq k < l \leq n+1, \\ \tau_{1l} &= \frac{a_n}{\sqrt{a_n^2 + b_n^2 d_n^2}} \lambda_{3n+1+l}, \quad 2 \leq l \leq n+1, & \tau_{kl} &= \lambda_{\bar{k}\bar{l}}, \quad 2 \leq k < l \leq n+1. \end{aligned}$$

Assume $\lambda_{2n+3} \neq 0$, then

$$\frac{\varsigma_{12}}{\xi_2} = -\frac{a_n}{b_n d_n}. \quad (4.5)$$

If $X \in \mathcal{C}F$, we have

$$\varsigma_{12} = \frac{\sqrt{|\xi|^2 + |\eta|^2}}{a_n} b_n d_n \operatorname{Re}(z_1 \bar{z}_2) = \frac{b_n d_n}{a_n \sqrt{|\xi|^2 + |\eta|^2}} \xi_1 \xi_2. \quad (4.6)$$

Combining (4.5) and (4.6), we obtain

$$\frac{\xi_1}{\sqrt{|\xi|^2 + |\eta|^2}} = -\frac{a_n^2}{b_n^2 d_n^2} \quad (4.7)$$

which implies

$$a_n - b_n \lambda_{2n+2} < 0. \quad (4.8)$$

On the other hand, from Lemma 3.1 and (4.7), we have

$$\begin{aligned} b_n + a_n \lambda_{2n+2} = \mu_1 &= \frac{\sqrt{|\xi|^2 + |\eta|^2}}{a_n} b_n c_{n,1} \left[\frac{\xi_1^2}{|\xi|^2 + |\eta|^2} - \frac{1}{n} \sum_{k=2}^{n+1} \frac{\xi_k^2 + \eta_k^2}{|\xi|^2 + |\eta|^2} \right] \\ &= \frac{\sqrt{|\xi|^2 + |\eta|^2}}{a_n} b_n c_{n,1} \left[\frac{\xi_1^2}{|\xi|^2 + |\eta|^2} - \frac{1}{n} \left(1 - \frac{\xi_1^2}{|\xi|^2 + |\eta|^2} \right) \right] \\ &= \frac{\sqrt{|\xi|^2 + |\eta|^2}}{a_n} b_n c_{n,1} \left[\frac{n+1}{n} \frac{1}{(2n+3)^2} - \frac{1}{n} \right] < 0. \end{aligned} \quad (4.9)$$

So (4.8) and (4.9) together lead to a contradiction. Thus $\lambda_{2n+3} = 0$.

Similarly, one can show $\lambda_{2n+1+l} = \lambda_{3n+1+k} = 0$ for $3 \leq l \leq n+1$ and $2 \leq k \leq n+1$. Consequently, $\xi_k = \eta_k = 0$ for $2 \leq k \leq n+1$. Note that $X, P \in \mathcal{C}F$ and their first $(2n+2)$ components form parallel vectors. By the geometric structure of $\mathcal{C}F$, this implies that $\vec{OX} \parallel \vec{OP}$. Since $\vec{OX} = \vec{OP} + \vec{PX}$ with $\vec{OP} \perp \vec{PX}$, it follows that $\vec{PX} = \vec{0}$, i.e., $X = P$. Hence the normal radius of $\mathcal{C}F$ is pointwise at least $\frac{\pi}{2}$.

In order to have estimates (2) on vanishing angles, we need to figure out \mathcal{S} for our cases. Suppose

$$\begin{aligned} v_0 &= \lambda_{2n+2} e_{2n+2} + \sum_{k=2}^{n+1} \left(\lambda_{2n+1+k} e_{2n+1+k} + \lambda_{3n+1+k} e_{3n+1+k} \right) \\ &\quad + \sum_{\alpha=2}^n \lambda_{4n+1+\alpha} e_{4n+1+\alpha} + \sum_{2 \leq k < l \leq n+1} \left(\lambda_{kl} e_{kl} + \lambda_{\bar{k}\bar{l}} e_{\bar{k}\bar{l}} \right) \end{aligned}$$

is a unit normal vector, such that

$$\| (h_{AB}^{v_0}) \| = \mathcal{S} = \max_{\mu \in T_P^\perp \mathcal{C}F, |\mu|=1} \| (h_{AB}^\mu) \|.$$

Let I be the set of indices of normal basis (3.4). According to behaviors of second fundamental form in normal directions, we split I into two parts:

$$\mathcal{A} = \{2n+2\} \cup \{4n+1+\alpha : 2 \leq \alpha \leq n\} \quad \text{and} \quad \mathcal{B} = I - \mathcal{A}.$$

Then, by Proposition 3.2,

- (★1) If $\tau \in \mathcal{A}$, then (h_{AB}^τ) is purely diagonal, and
- (★2) If $\tau \in \mathcal{B}$, then $\{h_{AA}^\tau\}$ are all zero. Moreover, for different $\tau, \mu \in \mathcal{B}$, $h_{AB}^\tau \cdot h_{AB}^\mu = 0$, $\forall A, B$.

By direct computations, we have

$$\| (h_{AB}^{2n+2}) \|^2 = \frac{n(2n+3)}{2(n+1)} + \frac{2n+3}{4(n+1)} = \frac{(2n+1)(2n+3)}{4(n+1)}$$

$$\|(h_{AB}^{2n+1+k})\|^2 = \|(h_{AB}^{3n+1+k})\|^2 = \frac{(2n+1)(2n+3)}{4(n+1)(n+2)} \quad \text{for } 2 \leq k \leq n+1$$

$$\|(h_{AB}^{4n+1+\alpha})\|^2 = \frac{(2n+1)(2n+3)}{2(n+1)(n+2)} \quad \text{for } 2 \leq \alpha \leq n$$

$$\|(h_{AB}^{(k,l)})\|^2 = \|(h_{AB}^{(\bar{k},\bar{l})})\|^2 = \frac{(2n+1)(2n+3)}{2(n+1)(n+2)} \quad \text{for } 2 \leq k < l \leq n+1$$

We shall show that $v_0 = \pm e_{2n+2}$ and $\mathcal{S}^2 = \frac{(2n+1)(2n+3)}{4(n+1)}$. The procedure consists of two steps.

Step 1: $\lambda_\tau = 0$ for $\tau \in \mathcal{B}$. Write $v_0 = c \cdot v_0^{\mathcal{A}} + s \cdot v_0^{\mathcal{B}}$ where $v_0^{\mathcal{A}}, v_0^{\mathcal{B}}$, if not zero, are the normalized unit vectors of projections of v_0 in $\text{span}\{e_\tau \mid \tau \in \mathcal{A}\}$ and $\text{span}\{e_\tau \mid \tau \in \mathcal{B}\}$ respectively, and where c, s stand for $\cos t$ and $\sin t$ for some real number t . Then

$$\|(h_{AB}^{v_0})\|^2 = \|(c \cdot h_{AB}^{v_0^{\mathcal{A}}} + s \cdot h_{AB}^{v_0^{\mathcal{B}}})\|^2 = c^2 \cdot \|(h_{AB}^{v_0^{\mathcal{A}}})\|^2 + s^2 \cdot \|(h_{AB}^{v_0^{\mathcal{B}}})\|^2$$

By $(\star 2)$, it follows $\|(h_{AB}^{v_0^{\mathcal{B}}})\|^2 \leq \frac{(2n+1)(2n+3)}{2(n+1)(n+2)}$. Now, if $\|(h_{AB}^{v_0^{\mathcal{A}}})\|^2$ were bounded from the above by the same number, then

$$\|(h_{AB}^{v_0})\|^2 \leq \frac{(2n+1)(2n+3)}{2(n+1)(n+2)} < \|(h_{AB}^{2n+2})\|^2$$

contradicting with our choice of v_0 . Hence,

$$\|(h_{AB}^{v_0^{\mathcal{B}}})\|^2 \leq \frac{(2n+1)(2n+3)}{2(n+1)(n+2)} < \|(h_{AB}^{v_0^{\mathcal{A}}})\|^2$$

and consequently

$$\|(h_{AB}^{v_0})\|^2 \leq \|(h_{AB}^{v_0^{\mathcal{A}}})\|^2 \Rightarrow \|(h_{AB}^{v_0})\|^2 = \|(h_{AB}^{v_0^{\mathcal{A}}})\|^2 \Rightarrow v_0 = v_0^{\mathcal{A}}, \text{ i.e. } \lambda_\tau = 0 \quad \text{for } \tau \in \mathcal{B}.$$

Step 2: $\lambda_\tau = 0$ for $\tau \in \mathcal{A} - \{2n+2\}$. We shall deduce the statement by induction in a reversed order. Write $v_0 = c \cdot E_{5n+1} + s \cdot e_{5n+1}$ where $s = \sin t = \lambda_{5n+1}$ for some $|t| \leq \frac{\pi}{2}$, and where E_{5n+1} is the normalized unit vector of $v_0 - \lambda_{5n+1} \cdot e_{5n+1}$ (nonzero, otherwise contradicting with the choice of v_0). Note that the nonzero elements of (h_{AB}^{5n+1}) are

$$-h_{nn}^{5n+1} = h_{n+1\ n+1}^{5n+1} = -h_{2n\ 2n}^{5n+1} = h_{2n+1\ 2n+1}^{5n+1} = -\frac{2b_n c_{n,n}}{a_n^2 + b_n^2 d_n^2},$$

and meanwhile, by $(\star 1)$ and Proposition 3.2, that all nonzero elements of $(h_{AB}^{E_{5n+1}})$ distribute in its diagonal and

$$h_{nn}^{E_{5n+1}} = h_{n+1\ n+1}^{E_{5n+1}} = h_{2n\ 2n}^{E_{5n+1}} = h_{2n+1\ 2n+1}^{E_{5n+1}}.$$

These nice distributions support

$$\|(h_{AB}^{v_0})\|^2 = \|(c \cdot h_{AB}^{E_{5n+1}} + s \cdot h_{AB}^{5n+1})\|^2 = c^2 \cdot \|(h_{AB}^{E_{5n+1}})\|^2 + s^2 \cdot \|(h_{AB}^{5n+1})\|^2. \quad (4.10)$$

By the same argument in Step 1, it follows that $s = \lambda_{5n+1} = 0$.

Assume that $\lambda_{5n+1} = \lambda_{5n} = \dots = \lambda_{4n+r+1} = 0$ for some $3 \leq r \leq n$. We aim to have $\lambda_{4n+r} = 0$. Similarly, write $v_0 = c \cdot E_{4n+r} + s \cdot e_{4n+r}$ where $s = \sin t = \lambda_{4n+r}$ for some $|t| \leq \frac{\pi}{2}$, and where $E_{4n+r} = k_1 e_{2n+2} + \sum_{i=2}^{r-2} k_i e_{4n+i+1}$ is the normalized unit vector of $v_0 - \lambda_{4n+r} \cdot e_{4n+r}$. Observe, from Proposition 3.2, that

$$(h_{AB}^{2n+2}) = \text{diag} \left(\frac{b_n}{a_n}, -\frac{(n+2)a_n b_n}{n(a_n^2 + b_n^2 d_n^2)}, \dots, -\frac{(n+2)a_n b_n}{n(a_n^2 + b_n^2 d_n^2)} \right), \quad (4.11)$$

and that, for $2 \leq \alpha \leq n$,

$$\begin{aligned} & (h_{AB}^{4n+\alpha+1}) \quad \quad \quad \alpha\text{-th} \\ & \quad \quad \quad \downarrow \\ & = \text{diag} \left(0, 0, \dots, 0, \frac{2b_n c_{n,\alpha}}{a_n^2 + b_n^2 d_n^2}, -\frac{2b_n c_{n,\alpha}}{(n+1-\alpha)(a_n^2 + b_n^2 d_n^2)}, \dots, -\frac{2b_n c_{n,\alpha}}{(n+1-\alpha)(a_n^2 + b_n^2 d_n^2)}, \right. \\ & \quad \quad \quad \left. 0, \dots, 0, \frac{2b_n c_{n,\alpha}}{a_n^2 + b_n^2 d_n^2}, -\frac{2b_n c_{n,\alpha}}{(n+1-\alpha)(a_n^2 + b_n^2 d_n^2)}, \dots, -\frac{2b_n c_{n,\alpha}}{(n+1-\alpha)(a_n^2 + b_n^2 d_n^2)} \right), \end{aligned} \quad (4.12)$$

where the first row includes $n+1$ elements and the second n terms. It follows

$$\begin{aligned} & (h_{AB}^{E_{4n+r}}) \quad \quad \quad (\mathbf{r-1})\text{-th} \\ & \quad \quad \quad \downarrow \\ & = \text{diag} \left(k_1 \frac{b_n}{a_n}, -\frac{k_1(n+2)a_n b_n}{n(a_n^2 + b_n^2 d_n^2)} + \frac{2k_2 b_n c_{n,2}}{a_n^2 + b_n^2 d_n^2}, \dots, *, \otimes, \otimes, \dots, \otimes, \right. \\ & \quad \quad \quad \left. -\frac{k_1(n+2)a_n b_n}{n(a_n^2 + b_n^2 d_n^2)} + \frac{2k_2 b_n c_{n,2}}{a_n^2 + b_n^2 d_n^2}, \dots, *, \otimes, \otimes, \dots, \otimes \right), \end{aligned} \quad (4.13)$$

where all \otimes s represent the same number. Hence,

$$\begin{aligned} (h_{AB}^{v_0}) &= \text{diag} \left(\left[k_1 \frac{b_n}{a_n} \right] c, \right. \\ & \quad \dots, [*]c, \otimes c + \left[\frac{2b_n c_{n,r-1}}{a_n^2 + b_n^2 d_n^2} \right] s, \otimes c - \left[\frac{2b_n c_{n,r-1}}{(n+2-r)(a_n^2 + b_n^2 d_n^2)} \right] s, \dots, \otimes c - \left[\frac{2b_n c_{n,r-1}}{(n+2-r)(a_n^2 + b_n^2 d_n^2)} \right] s, \\ & \quad \left. \dots, [*]c, \otimes c + \left[\frac{2b_n c_{n,r-1}}{a_n^2 + b_n^2 d_n^2} \right] s, \otimes c - \left[\frac{2b_n c_{n,r-1}}{(n+2-r)(a_n^2 + b_n^2 d_n^2)} \right] s, \dots, \otimes c - \left[\frac{2b_n c_{n,r-1}}{(n+2-r)(a_n^2 + b_n^2 d_n^2)} \right] s \right). \end{aligned} \quad (4.14)$$

Consequently,

$$\begin{aligned} & \| (h_{AB}^{v_0}) \|^2 \\ &= c^2 \cdot \| (h_{AB}^{E_{4n+r}}) \|^2 + s^2 \cdot \| (h_{AB}^{4n+r}) \|^2 + 4cs \cdot \left[\left(\frac{2b_n c_{n,r-1}}{a_n^2 + b_n^2 d_n^2} \right) \otimes - (n+2-r) \frac{2b_n c_{n,r-1}}{(n+2-r)(a_n^2 + b_n^2 d_n^2)} \otimes \right] \\ &= c^2 \cdot \| (h_{AB}^{E_{4n+r}}) \|^2 + s^2 \cdot \| (h_{AB}^{4n+r}) \|^2. \end{aligned} \quad (4.15)$$

Similarly, repeating the argument in Step 1 confirms that $\lambda_{4n+r} = 0$. Then, by induction, v_0 has to be $\pm e_{2n+2}$.

Now we figure out $\mathcal{S}^2 = \frac{(2n+1)(2n+3)}{4(n+1)} < n+1 = \frac{1}{2} \dim(\mathcal{C}F)$. It can be seen from Lawlor's table that, for $2 \leq n \leq 5$, vanishing angle θ_0 exists and is less than 45° . For $n > 5$, namely $m = \dim(\mathcal{C}F) = 2n+2 > 12$, by (4.3) and (4.4) we have

$$\tan\left(V(m, \sqrt{\frac{m}{2}})\right) = \tan\left(V(m, \frac{m}{12} \sqrt{\frac{72}{m}})\right) < \frac{12}{m} \tan\left(V(12, \sqrt{\frac{72}{m}})\right) < \tan\left(V(12, \sqrt{6})\right) \approx 8.36^\circ. \quad (4.16)$$

Hence, θ_0 exists as well. Combined with the result in [HL82] (about the classical coassociative Lawson-Osserman cone in \mathbf{R}^7 when our $n=1$), the proof completes. \square

Remarks. (a) We would like to point out that the coassociative cone's being area-minimizing cannot be verified following Lawlor's curvature criterion. Note that

$$\det(I - t(h_{AB}^4)) \geq \inf_{v \in T_p^\perp \Sigma, |v|=1} \left(\det(I - t(h_{AB}^v)) \right) \geq F\left(\sqrt{\frac{15}{8}}, t, 3\right) \quad (4.17)$$

where $F(\cdot, \cdot, \cdot)$ is the control which Lawlor used for his table of vanishing angles for $\dim < 12$. In the concrete case equalities are attained for all t but the case $(\dim, \mathcal{S}^2) = (4, \frac{15}{8})$ supports no vanishing angle by numerical computation.

(b) Type II enjoys similar properties as Type I. From (3.6), we get

$$\begin{aligned} \|h_{AB}^{4n+4}\|^2 &= \frac{n(4n+5)}{2(n+1)} + \frac{3(4n+5)}{8(n+1)} = \frac{(4n+3)(4n+5)}{8(n+1)} \\ \|h_{AB}^{4n+1+2k}\|^2 &= \|h_{AB}^{4n+2+2k}\|^2 = \frac{3(4n+3)(4n+5)}{16(n+1)(n+2)} \quad \text{for } 2 \leq k \leq n+1 \\ \|h_{AB}^{6n+1+2k}\|^2 &= \|h_{AB}^{6n+2+2k}\|^2 = \frac{(4n+3)(4n+5)}{4(n+1)(n+2)} \quad \text{for } 2 \leq k \leq n+1 \\ \|h_{AB}^{8n+3+\alpha}\|^2 &= \frac{(4n+3)(4n+5)}{4(n+1)(n+2)} \quad \text{for } 2 \leq \alpha \leq n \\ \|h_{AB}^{(k,l)}\|^2 &= \|h_{AB}^{(\bar{k}, \bar{l})}\|^2 = \|h_{AB}^{[k,l]}\|^2 = \|h_{AB}^{[\bar{k}, \bar{l}]}\|^2 = \frac{(4n+3)(4n+5)}{4(n+1)(n+2)} \quad \text{for } 2 \leq k < l \leq n+1 \end{aligned}$$

The same idea shows that $\mathcal{S}^2 = \frac{(4n+3)(4n+5)}{8(n+1)} < \frac{1}{2} \dim \mathcal{C}F'$ and therefore that $\mathcal{C}F'$ are area-minimizing for all $n \geq 1$.

(c) A similar calculation for Type III gives $\mathcal{S}^2 = \frac{51}{4} + \frac{119}{32} < 17$. By Lawlor's table of vanishing angles, (4.3) and (4.4), it follows that $V(16, \mathcal{S}) < V(16, \sqrt{17}) < V(12, \sqrt{17}) \approx 10.11^\circ < \frac{\pi}{4}$. So, $\mathcal{C}F''$ turns out to be area-minimizing as well.

Acknowledgments. This work was supported by NSFC (Grant No. 11471299, 11471078, 11622103, 11526048, 11601071), the Fundamental Research Funds for the Central Universities, and the SRF for ROCS, SEM.

References

- [Alm66] Frederick J. Almgren, *Some interior regularity theorems for minimal surfaces and an extension of Bernstein's theorem*, Ann. Math. **84** (1966), 277–292.
- [BDGG69] Enrico Bombieri, Ennio De Giorgi, and E. Giusti, *Minimal cones and the Bernstein problem*, Invent. Math. **7** (1969), 243–268.
- [dCW71] Manfredo P. do Carmo and Nolan R. Wallach, *Minimal immersions of spheres into spheres*, Ann. Math. **93** (1971), 43–62.
- [Che88] Benny N. Cheng, *Area-minimizing cone-type surfaces and coflat calibrations*, Indiana Univ. Math. J. **37** (1988), 505–535.
- [Fed69] Herbert Federer, *Geometric Measure Theory*, Springer-Verlag, New York, 1969.
- [DG65] Ennio De Giorgi, *Una estensione del teorema di Bernstein*, Ann. Scuola Norm. Sup. Pisa **19** (1965), 79–85.
- [FK85] Dirk Ferus and Hermann Karcher, *Non-rotational minimal spheres and minimizing cones*, Comment. Math. Helv. **60** (1985), 247–269.
- [Fle62] Wendell H. Fleming, *On the oriented Plateau problem*, Rend. Circ. Mat. Palermo **11** (1962), 69–90.
- [HS85] Robert Hardt and Leon Simon, *area-minimizing hypersurfaces with isolated singularities*, J. Reine. Angew. Math. **362** (1985), 102–129.
- [HL82] F. Reese Harvey and H. Blaine Lawson, Jr., *Calibrated geometries*, Acta Math. **148** (1982), 47–157.
- [KL99] Michael Kerckhove and Gary R. Lawlor, *A family of stratified area-minimizing cones*, Duke Math. J. **96** (1999), 401–424.
- [Law91] Gary R. Lawlor, *A Sufficient Criterion for a Cone to be Area-Minimizing*, Vol. 91, Mem. of the Amer. Math. Soc., 1991.
- [Law72] H. Blaine Lawson, Jr., *The equivariant Plateau problem and interior regularity*, Trans. Amer. Math. Soc. **173** (1972), 231–249.
- [LO77] H. Blaine Lawson, Jr. and Robert Osserman, *Non-existence, Non-uniqueness and Irregularity of Solutions of the Minimal Surface System*, Acta Math. **139** (1977), 1–17.
- [Mas80] Katsuya Mashimo, *Degree of the standard isometric minimal immersions of complex projective spaces into spheres*, Tsukuba J. Math. **4** (1980), 133–145.
- [Sim74] Plinio Simoes, *On a class of minimal cones in \mathbb{R}^n* , Bull. Amer. Math. Soc. **3** (1974), 488–489.
- [Sim73] ———, *A class of minimal cones in \mathbb{R}^n , $n \geq 8$, that minimize area*, Ph.D. thesis, University of California, Berkeley, Calif., 1973.
- [Sim83] Leon Simon, *Lectures on Geometric Measure Theory*, Vol. 3, Proc. Centre Math. Anal. Austral. Nat. Univ., 1983.
- [Sim68] James Simons, *Minimal varieties in riemannian manifolds*, Ann. Math. **88** (1968), 62–105.
- [Tak66] Tsunero Takahashi, *Minimal immersions of Riemannian manifolds*, J. Math. Soc. Japan **18** (1966), 380–385.

- [Tan01] Zizhou Tang, *Nonexistence of a submersion from the 23-sphere to the Cayley projective plane*, Bull. London Math. Soc. **No.3(33)** (2001), 347-350.
- [Tot97] Gabor Toth, *Eigenmaps and the space of minimal immersions between spheres*, Indiana Univ. Math. J. **46** (1997), 637–658.
- [Ura85] Hajime Urakawa, *Minimal immersions of projective spaces into spheres*, Tsukuba J. Math. **9** (1985), 321–347.
- [XYZ] Xiaowei Xu, Ling Yang, and Yongsheng Zhang, *On the Lawson-Osserman constructions*, preprint.

Xiaowei Xu,
 School of Mathematical Sciences, University of Science and Technology of China, Hefei, 230026,
 Anhui province, China;
 and Wu Wen-Tsun Key Laboratory of Mathematics, USTC, Chinese Academy of Sciences,
 Hefei, 230026, Anhui, P.R. China.
 E-mail: xwxu09@ustc.edu.cn.

Ling Yang,
 School of Mathematical Sciences, Fudan University, Shanghai 200433, P.R. China.
 E-mail: yanglingfd@fudan.edu.cn

Yongsheng Zhang,
 School of Mathematics and Statistics, Northeast Normal University, ChangChun 130024, Jilin
 province, P.R. China
 E-mail: yongsheng.chang@gmail.com